Comparison of Finite Difference Methods on One Dimensional Heat Equation

Swe Zin Nwe [*]

Abstract

Numerical techniques are powerful tools for solving the partial differential equations. A few problems can be solved analytically as well as difficult boundary value problems can be solved by numerical methods easily. A numerical method known as finite difference methods (explicit, fully implicit and Crank–Nicolson schemes) is applied for solving the heat equations successfully. In this paper, the solutions of finite difference methods are presented in tables together with figures comparing the analytical solution.

Keywords: Finite difference methods, Local truncation error, boundary condition, stability and convergence.

1. Introduction

The most common finite difference representation of derivative is based on Taylor's series expansions. The heat equation is fundamental in scientific fields. There are derivatives with respect to time and derivatives with respect to space in the heat equation.

The main objective of this paper is to study the effect of explicit, fully implicit and Crank–Nicolson schemes on one dimensional heat equation with initial–boundary condition. In explicit scheme, U at all grid points at time level j + 1 are calculated from the known value at time level j by the boundary conditions. Subsequently, fully implicit scheme and Crank–Nicolson scheme are solved by using MATLAB. Moreover, the Crank–Nicolson scheme can be obtained from the average of the explicit and fully implicit schemes. The numerical results are compared with the analytical solution. Finally, convergence, stability and discretization errors are presented for different schemes.

[*] Lecturer, Dr., Department of Mathematics, Yadanabon University

2. The Concept of a Taylor Series

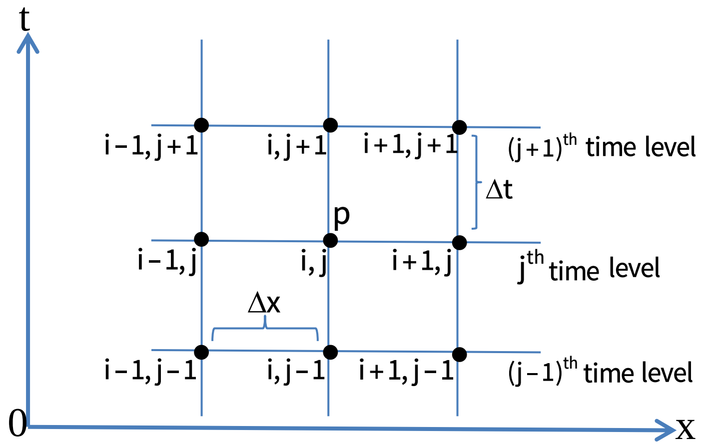

Figure1. Discrete grid points

If $u_{i,j}$ denote the t-component of velocity at point (i,j), then the velocity $u_{i,j+1}$ at point $(i,j+1)$ can be expressed in term of a Taylor series expanded about the point (i,j) as follow.

$$u_{i,j+1} = u_{i,j} + \left(\frac{\partial u}{\partial t}\right)_{i,j} \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} \frac{(\Delta t)^2}{2!} + \left(\frac{\partial^3 u}{\partial t^3}\right)_{i,j} \frac{(\Delta t)^3}{3!} + \cdots.$$

By Taylor's series expansion,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} + O(\Delta t),$$ (First-order forward difference in time)

$$\frac{\partial u}{\partial t} = \frac{u_{i,j} - u_{i,j-1}}{\Delta t} + O(\Delta t),$$ (First-order backward difference in time)

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} + O(\Delta t)^2$$ (Second-order Central difference in time)

and $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} + O(\Delta x)^2.$$ (second-order Central second difference in space)

The terms $O(\Delta t)$, $O(\Delta t)^2$ and $O(\Delta x)^2$ are denoted by the local truncation errors. When the local truncation errors are being neglected, finite difference schemes are obtained.

3. Explicit Scheme

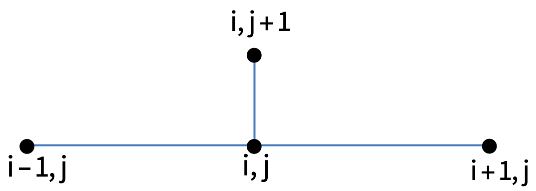

Figure 2. The explicit stencil

The one–dimensional heat equation is considered as:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}. \tag{1}$$

Using forward difference at time and a second order central difference in space, the following equation is obtained,

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}$$

which can be written as

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j} \tag{2}$$

where $r = \frac{\Delta t}{(\Delta x)^2}$.

This explicit method is known to be numerically stable and convergent whenever $r \le \frac{1}{2}$. The numerical errors are proportional to the time step and the square of the space step. In this method, the state of a system at a later time from the state of the system at the current time can be calculated. Therefore, explicit scheme is very simple.

4. Fully Implicit Scheme

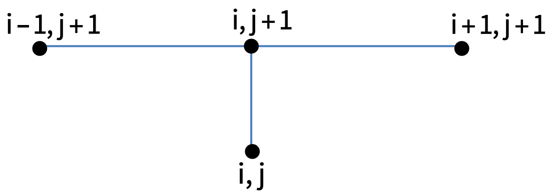

Figure 3. The fully implicit stencil

Substituting the backward difference at time and second order central difference in space, we get

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{(\Delta x)^2}.$$

We can obtain $u_{i,j+1}$ from solving above equation

$$-ru_{i-1,j+1} + (1+2r)u_{i,j+1} - ru_{i+1,j+1} = u_{i,j} \tag{3}$$

where $r = \frac{\Delta t}{(\Delta x)^2}$.

This scheme is called fully implicit scheme. The scheme is always numerically stable and convergent but usually more numerically intensive than the explicit scheme. The errors are linear over the time step and quadratic over the space step.

In order to obtain the function values at time step (j +1) need to solve a set of simultaneous linear equations (3), which can be cast in matrix form as

$$\begin{bmatrix} 1+2r & -r & 0 & 0 & 0 & 0 \\ -r & 1+2r & -r & 0 & 0 & 0 \\ 0 & -r & 1+2r & -r & 0 & 0 \\ 0 & 0 & -r & 1+2r & -r & 0 \\ 0 & 0 & 0 & -r & 1+2r & -r \\ 0 & 0 & 0 & 0 & -r & 1+2r \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \\ u_{5,j+1} \\ u_{6,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \\ u_{5,j} \\ u_{6,j} \end{bmatrix}.$$

5. Crank–Nicolson Scheme

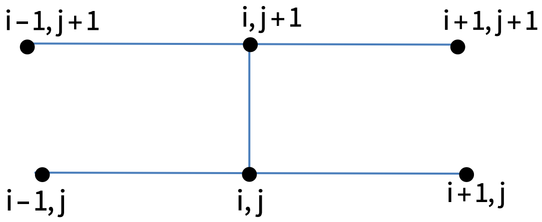

Figure 4. The Crank–Nicolson stencil

If $\frac{\partial U}{\partial t}$ is replaced by forward difference approximation and $\frac{\partial^2 U}{\partial x^2}$ by average of central difference in space at j and j+1 time level in (1),

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \right)$$ is obtained.

It can be written as

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j} \quad (4)$$

where $r = \frac{\Delta t}{(\Delta x)^2}$.

This scheme is always numerically stable and convergent. The errors are quadratic over the time step as well as the space step. The Crank–Nicolson method is combined from the average of the explicit and fully implicit schemes.

5.1 Example

The heat equation $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ is considered with the initial condition $U(x,0) = \sin\pi x$, $0 \le x \le 1$ and boundary conditions are $U(0,t) = U(1,t) = 0$.

By solving this problem, it is easily obtained that the exact solution of the problem is $U(x,t) = e^{-\pi^2 t}\sin\pi x$.

It is considered as $\Delta x = 0.1$, $\Delta t = 0.005$.

So $r = \dfrac{\Delta t}{(\Delta x)^2} = \dfrac{0.005}{0.01} = 0.5$.

Therefore, stability condition of explicit finite scheme is satisfied and a stable condition is expected.

By using boundary condition, $u_{0,j} = 0, u_{0,j+1} = 0$ and $u_{n,j} = 0, u_{n,j+1} = 0$ are obtained.

Using the initial condition when t=0 and $0 \le x \le 1$, it is obtained as

$u_{0,0} = \sin 0 = 0$, $u_{1,0} = \sin(\pi \times 0.1) = 0.3090$, $u_{2,0} = \sin(\pi \times 0.2) = 0.5878$, $u_{3,0} = \sin(\pi \times 0.3) = 0.8090$, $u_{4,0} = \sin(\pi \times 0.4) = 0.9511$, $u_{5,0} = \sin(\pi \times 0.5) = 1$.

By symmetry, $u_{6,0} = u_{4,0}$, $u_{7,0} = u_{3,0}$, $u_{8,0} = u_{2,0}$, $u_{9,0} = u_{1,0}$ and $u_{10,0} = u_{0,0}$.

By using the boundary and initial conditions, the solutions of explicit scheme can be found.

Substituting r = 0.5 in (2), the explicit method is

$$u_{i,j+1} = 0.5u_{i-1,j} + 0.5u_{i+1,j}.$$

For i=1, j=0, $u_{1,1} = 0.5u_{2,0} = 0.2939$,

for i=2, j=0, $u_{2,1} = 0.5u_{1,0} + 0.5u_{3,0} = 0.5590$,

for i=3, j=0, $u_{3,1} = 0.5u_{2,0} + 0.5u_{4,0} = 0.7694$,

for i=4, j=0, $u_{4,1} = 0.5u_{3,0} + 0.5u_{5,0} = 0.9045$,

for i=5, j=0, $u_{5,1} = 0.5u_{4,0} + 0.5u_{6,0} = 0.9511$,

for i=6,j=0, $u_{6,1} = 0.5u_{5,0} + 0.5u_{7,0} = 0.9045$.

The fully implicit tri-diagonal matrix that are substituted by r=0.5, j=0 in (3) is

$$\begin{bmatrix} 2 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 2 & -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 2 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 2 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 2 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 2 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ u_{5,1} \\ u_{6,1} \end{bmatrix} = \begin{bmatrix} 0.3090 \\ 0.5878 \\ 0.8090 \\ 0.9511 \\ 1.0000 \\ 0.9511 \end{bmatrix}.$$

The solutions of fully implicit scheme by solving above matrix are

$u_{1,1} = 0.2946$, $u_{2,1} = 0.5604$, $u_{3,1} = 0.7713$, $u_{4,1} = 0.9067$, $u_{5,1} = 0.9535$, $u_{6,1} = 0.9071$.

Substituting r = 0.5, j = 0 in (4), we get the Crank–Nicolson tri-diagonal matrix form as

$$\begin{bmatrix} 3 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 3 & -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 3 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 3 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 3 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 3 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ u_{5,1} \\ u_{6,1} \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.3090 \\ 0.5878 \\ 0.8090 \\ 0.9511 \\ 1.0000 \\ 0.9511 \end{bmatrix}$$

By solving the above matrix, the solutions of Crank–Nicolson scheme are

$u_{1,1} = 0.2943,\ u_{2,1} = 0.5597,\ u_{3,1} = 0.7704,\ u_{4,1} = 0.9056,\ u_{5,1} = 0.9523,\ u_{6,1} = 0.9057$.

Table 1 Comparison of Finite Difference Schemes at t = 0.005

x	Explicit	Fully Implicit	Crank–Nicolson	Exact
0.1	0.2939	0.2946	0.2942	0.2941
0.2	0.5590	0.5604	0.5597	0.5595
0.3	0.7694	0.7713	0.7704	0.7701
0.4	0.9045	0.9067	0.9056	0.9053
0.5	0.9511	0.9535	0.9523	0.9518
0.6	0.9045	0.9071	0.9057	0.9053

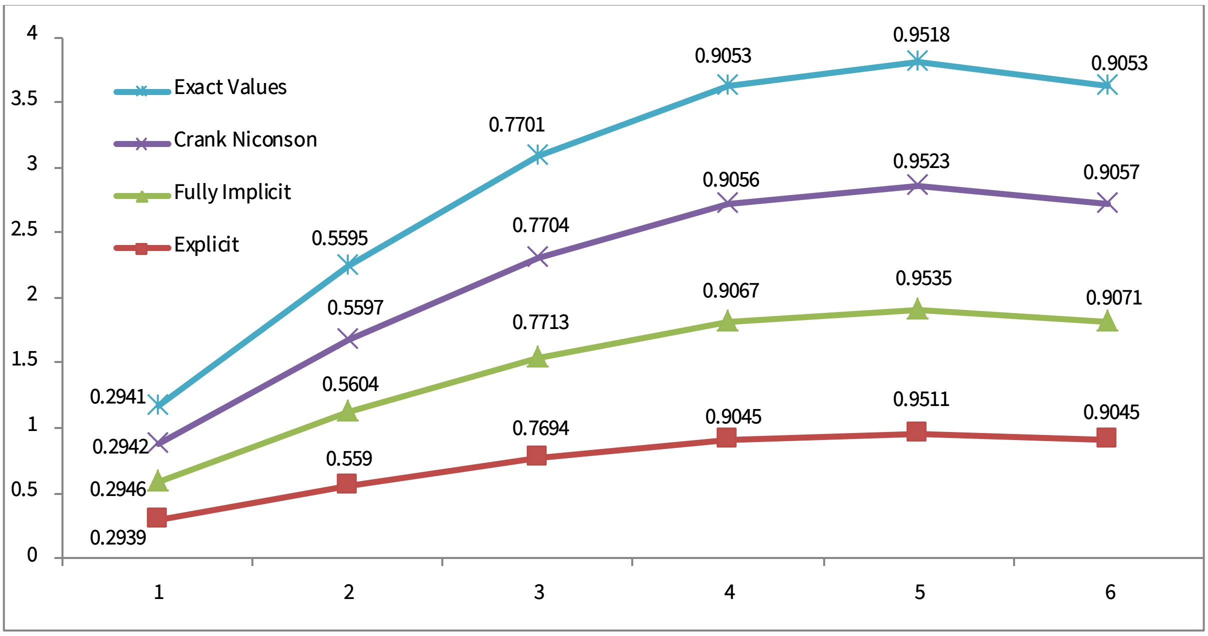

Figure 5. Comparison of finite difference schemes

6. Result and Discussion

Sometimes differential equations are very difficult to solve analytically or models are needed for computer simulations. In these cases finite difference methods are used to solve the equations instead of analytical ones.

Finite difference methods are numerical methods for solving differential equations by approximating them with difference equations in which finite differences approximate the derivatives.

For example, in electronics and electrical engineering the differential equations describing complex circuits containing capacitors, inductors and resistors can be replaced with finite difference equations. Computer simulations of the models are used to estimate voltage and currents in the nodes of the circuit.

Table 2 Percentage errors for finite difference schemes at t=0.005

x	Explicit % error	Fully Implicit % error	Crank–Nicolson % error
0.1	0.0680	0.1700	0.0340
0.2	0.0894	0.1609	0.0357
0.3	0.0909	0.1558	0.0390
0.4	0.0884	0.1786	0.0331
0.5	0.0736	0.1786	0.0525
0.6	0.0884	0.1988	0.0442

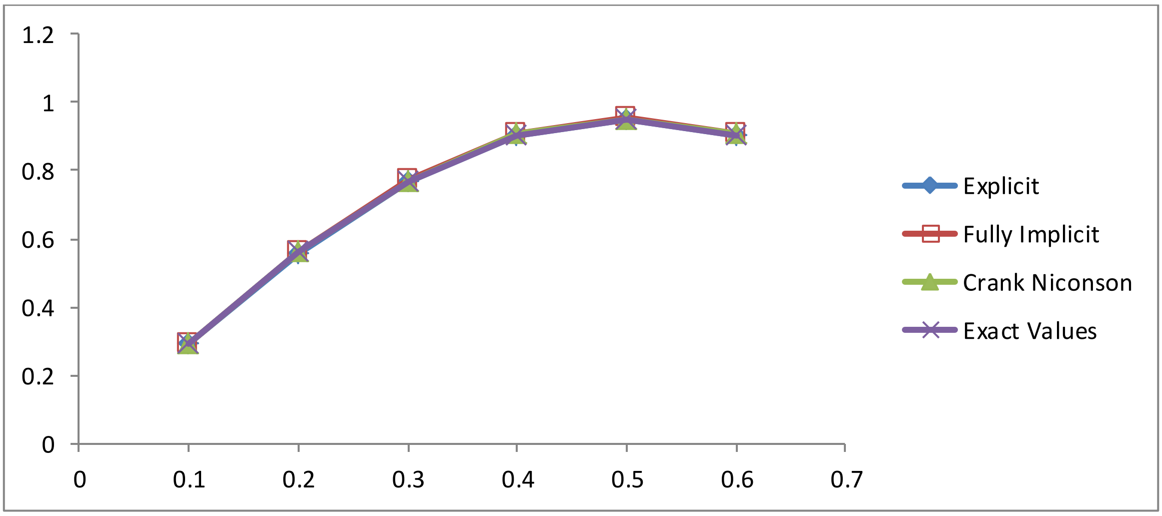

Figure 6. Convergence plots of Finite difference schemes

Conclusion

The aim of this paper is to compare finite difference schemes. Comparing with the analytical results, the best approximate solution is given from Crank-Nicolson method. Similarly, the solutions of wave equations and Laplace equations can be found by using Crank-Nicolson method.

Acknowledgements

I would like to pay my deep thanks to Dr. Tint Moe Thuzar, Rector, Dr. U Khin Myot, Pro-Rector, Dr. Khin Maw Maw Soe, Pro-Rector and Dr. Myint Myint Oo, Pro-Rector Yadanabon University for their exhorting and encouragement to submit this paper. And then, I am deeply indebted to Dr. Mon Yee Aye, Professor and Head of Department of Mathematics and Dr. Daw Cho, Professor, Department of Mathematics, Yadanabon University, for their invaluable advice and helpful comments on this research.

References

Anderson, J.D.Jr., “Computational Fluid Dynamics”, McGraw-Hill, Inc, New York, 1995.

Smit, G.D., "Numerical Solution of Partial Differential Equation: Finite Difference Methods", Clarendon Press, Oxford, 1985.

Tyn Myin-U., "Partial Differential Equations for Scientists and Engineers", North Holland, New York, Third Edition, 1987.